[redacted]

# Twenty-Sixth Annual Mathematics Tournament
## April 15, 2023
## Solutions

Round 1: Total number of tulip bulbs: 30, number of yellow tulip bulbs: 10, number of red tulip bulbs: 10. Let $R$ : a randomly selected tulip bulb is red, $Y$ : a randomly selected tulip bulb is yellow. So, we want to know $P(R_1 \cap Y_2) + P(Y_1 \cap R_2)$, where $R_i, Y_i$ stands for red tulip bulb or yellow tulip bulb in the $i^{\text{th}}$ draw, $i = 1, 2$.

$$P(R_1 \cap Y_2) + P(Y_1 \cap R_2) = P(Y_2 \mid R_1) \cdot P(R_1) + P(R_2 \mid Y_1) \cdot P(Y_1)$$
$$= \frac{10}{29} \cdot \frac{10}{30} + \frac{10}{29} \cdot \frac{10}{30} = \frac{10}{87} + \frac{10}{87} = \frac{20}{87}.$$

Round 2: Let $n(x)$ represent the number of elements in the set represented by $x$. Let $U$ be the universal set.

$$n(B12) = 150, n(C) = 200, n(E) = 165$$

$n(B12$

Round 3: $F$: original amount of grass grass in the field, $G$: amount of grass growing daily, $C$: amount of grass a cow eats daily

$$F + 14G = 14(60C) \Rightarrow F + 14G = 840C$$
$$F + 28G = 28(50C) \Rightarrow F + 28G = 1400C$$
$$14G = 560C \Rightarrow G = \frac{560}{14}C$$
$$\Rightarrow G = 40C$$

Hence, the maximum number of cows would be 40.

Round 4: Extend the sequence a little bit more and see the pattern.

$$3; 3; 2; 1; 3; 0; 3; 3; 2; 1; 3; 0 \ldots$$

So, we find a cycle of length 6. Dividing 1209 by 6, we will obtain the quotient of 201 with a remainder of 3. This represents 201 complete cycles of length 6 and 3 digits into the next cycle, which would give the digit 2 as the $1209^{\text{th}}$ term. Hence the $1209^{\text{th}}$ term is 2.

Round 5:

$$\text{Let } \log_9(p) = \log_{12}(q) = \log_{16}(p+q) = k.$$
$$\Rightarrow \log(p) = k\log 9 = 2k\log 3$$
$$\log(q) = k\log 12 = 2k\log 2 + k\log 3$$
$$\log(p+q) = k\log 16 = 4k\log 2.$$
$$\text{Hence, } \log(q) = \frac{1}{2}\log(p+q) + \frac{1}{2}\log p$$
$$\Rightarrow 2\log(q) = \log\left(p^2 + pq\right)$$
$$\Rightarrow q^2 = p^2 + pq \Rightarrow \left(\frac{q}{p}\right)^2 - \frac{q}{p} - 1 = 0$$
$$\Rightarrow \frac{q}{p} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$
$$\text{So, } \frac{q}{p} = \frac{1+\sqrt{5}}{2} \text{ as } \frac{q}{p} > 0.$$

Round 6: For $n^2 - 2n - 8$ to be equal to a prime number, one of its factors must be equal to 1. Factoring gives $(n-4)(n+2)$, so either $n - 4 = 1$ or $n + 2 = 1$. Solving the first equation gives $n = 5$, and solving the second equation gives $n = -1$. $n = -1$ is not a natural number, thus not an answer. $n = 5$ is a natural number and makes $(n-4)(n+2) = (5-4)(5+2) = (1)(7) = 7$, which is a prime number. $n = 5$ is the only natural number.

log

Round 7: Let $m; n$ be the numbers of digits in $2^{2005}$ and $5^{2005}$. Then observe that

$$10^{m-1} < 2^{2005} < 10^{m} \quad ; \quad 10^{n-1} < 5^{2005} < 10^{n}$$

implying that

$$10^{m+n-2} < 2^{2005} \quad 5^{2005} = 10^{2005} < 10^{m+n}:$$

Hence $m + n \quad 2 < 2005 < m + n$, so $m + n = 2006$, which is the answer.

Round 8: We wish to seat 5 people, 3 of whom must sit consecutively, in 12 seats. Since the block takes up 3 of the 12 places, it must begin in one of the rst $12 \quad (3 \quad 1) = 12 \quad 3 + 1 = 10$ positions. Once the block has been placed, there are $12 \quad 3 = 9$ seats left for the remaining $5 \quad 3 = 2$ people. They can be arranged in those seats in $P(9; 2)$ ways. The people within the block can be arranged in 3! ways, which gives us the formula $10 \quad P(9; 2) \quad 3! = 10 \quad 72 \quad 6 = 4320$ ways.

Note here $P(n; r)$ stands for the number of permutations of $r$ items out of a pack of $n$ items and is de ned via $P(n; r) = \dfrac{n!}{(n \quad r)!}$, where $0! = 1; 1! = 1; 2! = 2 \quad 1 = 2$, $3! = 3 \quad 2 \quad 1 = 6; 4! = 4 \quad 3 \quad 2 \quad 1 = 24$, and in general $n! = n \quad (n \quad 1) \quad 3 \quad 2 \quad 1$, $n$ is a non negative integer.

Round 9: $\log_2 \quad \log_4 \quad \log_{\frac{1}{2}}(\log_9(2k)) \quad = \quad 1$

$$) \log_4 \quad \log$$

9